Gravity as a mesoscopic system

Based on 2409.13808 with Julian Sonner and Herman Verlinde

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Universality in statistical physics

The physics of certain classes of observables of many-body systems is independent of the microscopic details of the theory

- Critical exponents of phase transitions

Universality in statistical physics

The physics of certain classes of observables of many-body systems is independent of the microscopic details of the theory

- Critical exponents of phase transitions
- Random Matrix Theory (RMT) universality

A quantum chaotic system behaves as a collection of random matrices that share the same set of discrete symmetries

  - Spectrum
  - Eigenvectors (ETH)

[Wigner 1956], [BGS 1984], [Altland, Zirnbauer 1997]

Universality in gravity

- No-hair theorem:

All stationary black-hole solutions of Einstein-Maxwell equations can be described by three independent parameters (M, Q and J).

[Israel 1967-8], [Carter 1971]

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- Gravity and RMT:
  - SFF for spectral correlations SYK: [CGHPSSSST 2018], JT-gravity: [SSS 2018], 3d-gravity: [CJ 2020]
  - ETH SYK: [SV 2018], JT-gravity: [Saad 2020], 3d-gravity: [BdB 2020], [CCHM 2022]

Gravity and stochastic processes

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- Main idea:
  - ‘Simple’ low energy observables, such as thermal correlation functions, behave as a stochastic process
  - Gravity computes moments of this stochastic process

Holography

- We consider gravity in (asymptotically) AdS_3
- We assume AdS/CFT [Maldacena 1997]
- Weakly coupled Einstein gravity is dual to a large-c CFT with a spectral gap and

$$c = \frac{3L}{2G_N}$$

[Brown, Henneaux 1986]

- Black holes are dual to thermal states in the CFT

Thermal correlation functions

- We consider the class of thermal correlation functions

$$G_\beta(t_1, \tau_1; \ldots; t_n, \tau_n) = \frac{1}{Z(\beta)} \operatorname{Tr}\left[e^{-\tau_n H}\, \mathcal{O}(t_n) \, \ldots \, e^{-\tau_1 H}\, \mathcal{O}(t_1) \right]$$

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- We specialise to the simplest one

$$G_\beta(t) = \frac{1}{Z(\beta)} \operatorname{Tr}\left[e^{-\beta H/2}\, \mathcal{O}(t)\, e^{-\beta H/2}\, \mathcal{O}(0)\right]$$

- Very natural set of observables

Thermal correlation functions in the bulk

- Action

$$I = \frac{1}{16\pi G_N} \int \mathrm{d}x^3 \sqrt{h} \left(R + \frac{2}{L^2} \right) + m \int \mathrm{d}l + S_{\mathrm{ct}}$$

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$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle \approx e^{-md(x,y)}$$

- The distance above is the length of the geodesics in the BTZ metric

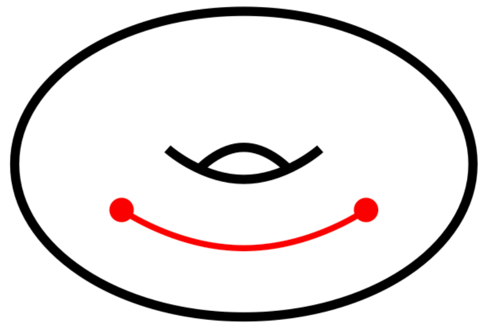
$$\mathrm{d}s^2 = -(r^2 - r_+^2)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2 - r_+^2} + r^2\,\mathrm{d}\theta^2$$

Finite volume!

Thermal correlation functions in the bulk

- Computing the geodesics in the bulk, the thermal two point function is

$$G_\beta(t) = \frac{\pi^{2\Delta}}{\beta^{2\Delta}} \cosh^{-2\Delta}\left(\frac{\pi t}{\beta}\right) + \dots$$



- Exponential decay!

[Maldacena 2001]

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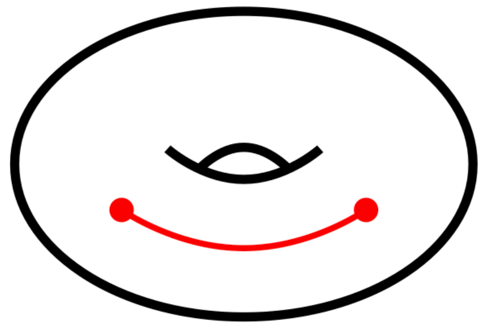

- Exponential decay!
- Corrections are geodesics that ‘wrap’ around the black hole, full solution is

$$G_\beta(t) = \left(\frac{2\pi}{\beta}\right)^{2\Delta} \sum_{n\in\mathbb{Z}} \frac{1}{\left[2\cosh(\frac{2\pi t}{\beta}) + 2\cosh(\frac{4\pi^2 n}{\beta})\right]^{\Delta}}$$

[Maldacena 2001]

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- Microscopic thermal correlation function reads

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- Riemann-Lebesgue lemma: it cannot vanish at late times
- Fluctuations at late times should be of the order $e^{-\tilde{c}S_\beta}$

[Maldacena 2001]

We propose a framework to interpret this result

Possible digression:
relation with [Saad 2020]

Brownian Motion

Particle suspended in a fluid

- If you want to solve the dynamics of a particle suspended in a fluid, the many-body problem becomes quickly intractable

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\sum_i V'(x - x_i) \qquad \frac{\mathrm{d}^2 x_i}{\mathrm{d}t^2} = -V'(x_i - x) - \sum_{i \neq j} V'_{\mathrm{VdW}}(x_i - x_j)$$

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- You can consider an effective theory, for example

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\gamma\, v(t) \qquad \longrightarrow \qquad v(t) = v_0\, e^{-\gamma t}$$

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- The result obtained is a good approximation of the dynamics for some time
- However, as before, it is incompatible with a microscopic interpretation, and with basic requirements of thermodynamics
- At late times, the particle and the fluid will have the same temperature
- Maxwell-Boltzmann distribution implies

$$\mathbb{E}[v^2(t)] \to \frac{k_B T}{m}$$

How can we take into account these fluctuations?

Brownian motion

- Idea: employ a probabilistic mesoscopic description, adding a stochastic noise

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\gamma\, v(t) + \boxed{\xi(t)} \qquad \text{with} \qquad \mathbb{E}[\xi(t)\xi(s)] = g\,\delta(t-s)$$

[Einstein 1905, Langevin 1908]

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- The parameter g sets the size of the fluctuations
- We are re-summing the effect of the interaction with the fluid particles into a deterministic plus stochastic term

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- The parameter g sets the size of the fluctuations
- We are re-summing the effect of the interaction with the fluid particles into a deterministic plus stochastic term
- This in turn makes the velocity a stochastic process itself

[Einstein 1905, Langevin 1908]

Brownian motion

- The first moment is unaffected

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathbb{E}[v(t)] = -\gamma\,\mathbb{E}[v(t)] \quad\longrightarrow\quad \mathbb{E}[v(t)] = v_0\,e^{-\gamma t}$$

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- The second moment is affected by connected correlations of the noise

$$\mathbb{E}[v(t)v(s)] = \mathbb{E}[v(t)]\,\mathbb{E}[v(s)] - \frac{g}{2\gamma}\,e^{-\gamma(t+s)} + \frac{g}{2\gamma}e^{-\gamma|t-s|}$$

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- This implies $\mathbb{E}[v^2(t)] \to \dfrac{g}{2\gamma}$ which gives $g = \dfrac{2\gamma k_B T}{m}$ (fluctuation-dissipation!)

[Langevin 1908]

Features of a mesoscopic description

Effective description	*Deterministic*	*Practical, but info loss*
Mesoscopic description	*Probabilistic*	*Practical, and 'less' info loss*
Microscopic description	*Deterministic*	*Impossible to solve*

When is a mesoscopic description effective?

Necessary conditions

- Everything we want to describe is fundamentally deterministic

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  1. A thermodynamic limit has to be performed
  2. There must be a clear scale separation between a probe and an environment
  3. The dynamics has to be observed for a long amount of time, much longer than the intrinsic fluctuations of the system

- Mean free time for a particle

- Inverse of spectral width for correlation functions

Back to holography

Gravity and moments

- Assume the result computed previously is an expectation value

$$\mathbb{E}[G_\beta(t)] = \frac{\pi^{2\Delta}}{\beta^{2\Delta}} \cosh^{-2\Delta}\left(\frac{\pi t}{\beta}\right)$$

Gravity and moments

- Assume the result computed previously is an expectation value

$$\mathbb{E}[G_\beta(t)] = \frac{\pi^{2\Delta}}{\beta^{2\Delta}} \cosh^{-2\Delta}\left(\frac{\pi t}{\beta}\right)$$

- Then, the analogy with stochastic processes suggests to compute

$$\mathbb{E}[G_\beta(t) G_\beta(s)]$$

- This is an observable that 'connects' two boundaries

Gravity and moments

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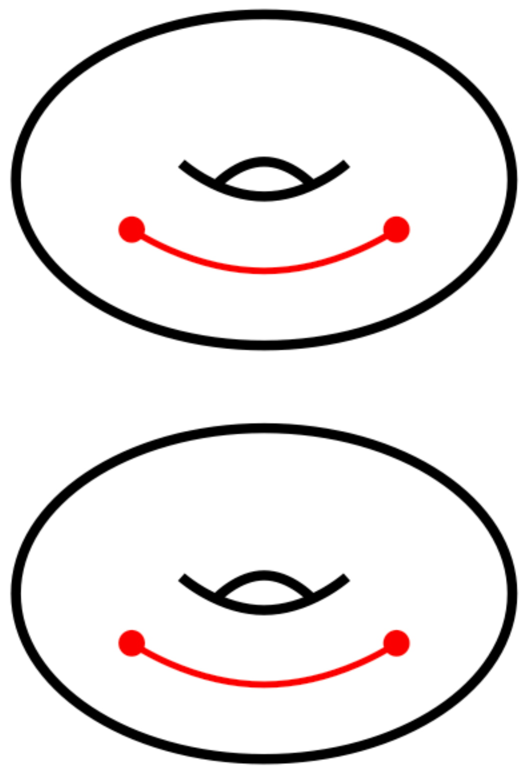
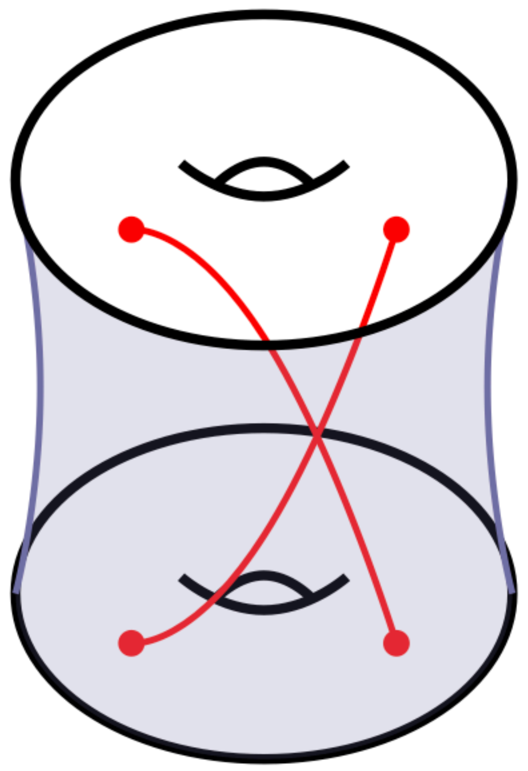
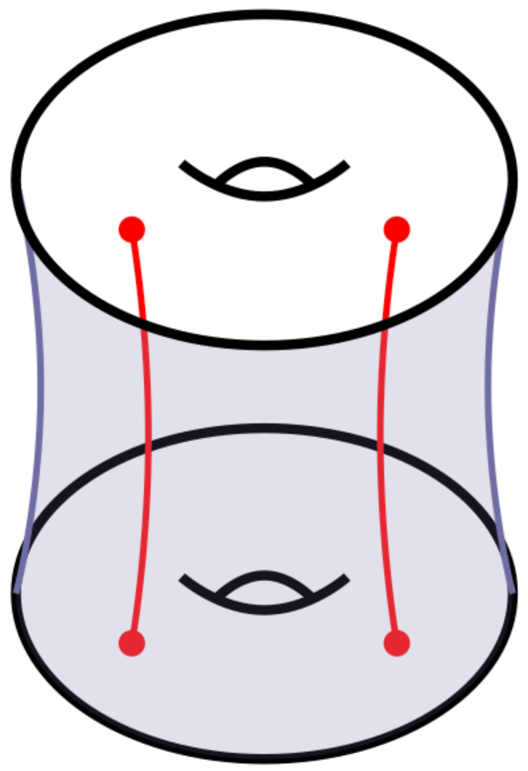
$\mathbb{E}[G_\beta(t)G_\beta(s)] =$  $+$  $+$  $\cdots$

Autocorrelation function

- Gravity computation has ‘simple’ answer in terms of Liouville correlators

[Chandra, Collier, Hartman, Maloney 2022]

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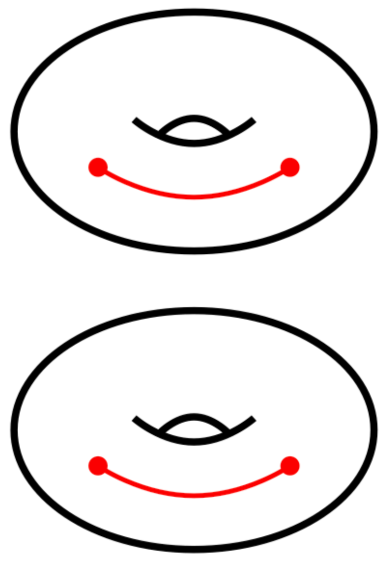
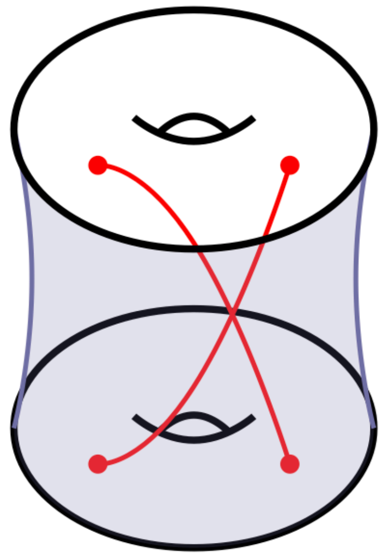
- Approximation at late times

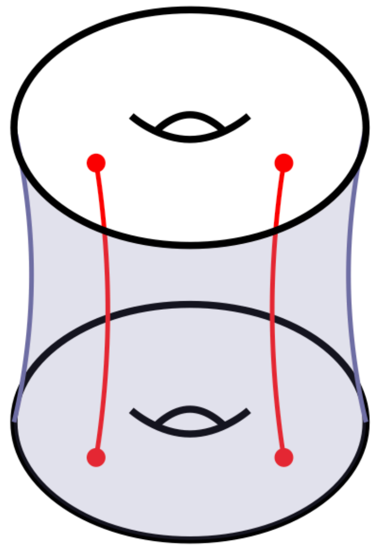
$$\mathbb{E}[G_\beta(t)G_\beta(s)] \approx$$



$$\mathbb{E}[G_\beta(t)]\,\mathbb{E}[G_\beta(s)]$$



$$+\ \frac{1}{Z^2(\beta)}\frac{\pi^{4\Delta}}{\beta^{4\Delta}}\cosh^{-2\Delta}\left(\frac{\pi t}{\beta}\right)\cosh^{-2\Delta}\left(\frac{\pi s}{\beta}\right)$$



$$+\ \frac{1}{Z^2(\beta)}\frac{\pi^{4\Delta}}{\beta^{4\Delta}}\cosh^{-2\Delta}\left(\frac{\pi(t-s)}{\beta}\right)$$

[Maldacena, Maoz 2004]

$$+\ \ldots$$

Comparison with Brownian motion

- Let's make a precise comparison
  - BM:

$$\mathbb{E}[v(t)v(s)] = \mathbb{E}[v(t)]\,\mathbb{E}[v(s)] - \frac{g}{2\gamma}\,e^{-\gamma(t+s)} + \frac{g}{2\gamma}e^{-\gamma|t-s|}$$

  - Gravity:

$$\mathbb{E}[G_\beta(t)G_\beta(s)] \approx \mathbb{E}[G_\beta(t)]\,\mathbb{E}[G_\beta(s)] + \frac{2^{4\Delta}\pi^{4\Delta}}{Z^2(\beta)\beta^{4\Delta}}e^{-\frac{2\pi\Delta(t+s)}{\beta}} + \frac{2^{2\Delta}\pi^{4\Delta}}{Z^2(\beta)\beta^{4\Delta}}e^{-\frac{2\pi\Delta|t-s|}{\beta}}$$

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Back to Brownian motion

- This approximate match becomes exact in the limit

$$\Delta \to 0 \;, \qquad \beta \to 0 \;, \qquad \text{fixing} \quad \gamma \equiv \frac{2\pi\Delta}{\beta}$$

knowing also the limit

$$\lim_{\Delta\to 0} \cosh\left(\frac{\gamma\, x}{2\Delta}\right)^{-2\Delta} = e^{-\gamma\,|x|}$$

and identifying $\qquad v_0 \equiv \dfrac{\pi^{2\Delta}}{\beta^{2\Delta}} \quad , \qquad \dfrac{g}{2\gamma} \equiv \dfrac{\pi^{4\Delta}}{Z^2(\beta)\,\beta^{4\Delta}}$

Late-time fluctuations of thermal correlators in gravity behave as Brownian motion!

A recursive structure

- The autocorrelation does not have complete microscopic information!
- When $t - s$ is large, there is still an exponential decay, since the noise becomes uncorrelated

$$\mathbb{E}[G_\beta^2(t)]\mathbb{E}[G_\beta^2(s)]$$

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- In order to be consistent with unitarity, we are led to consider higher moments, such as

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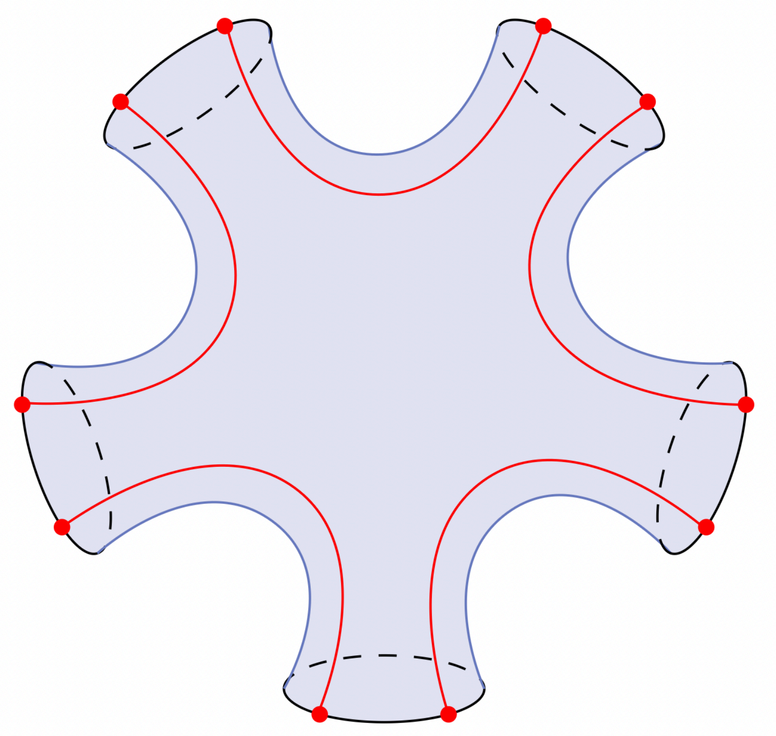

- Multi-boundary wormholes, non-Gaussian corrections!

A summary up to this point

- A mesoscopic description of a system is an intermediate description that contains information on the microscopic theory *in a probabilistic sense*
- Thermal correlation functions in semiclassical gravity in AdS are consistently described by a mesoscopic description
    - The 'naive' (trivial topology) answer is the *mean* of the process
    - Wormholes represent *higher moments* of the process
    - In a suitable double scaling limit (and up to two-day wormholes), the stochastic process is Brownian motion

Microscopic interpretation

Eigenstate Thermalisation Hypothesis

- Developed to explain thermalisation in closed systems
- Consider $|\Psi\rangle = \sum_i c_i \, |E_i\rangle$

$$\langle \mathcal{O}(t) \rangle_\Psi = \sum_{ij} c_i^* c_j \, \mathcal{O}_{ij} \, e^{-i(E_j - E_i)t} \quad \to \quad \sum_i |c_i|^2 \mathcal{O}_{ii}$$

Thermal expectation?

- Two problems
  1. It depends on the initial state through the c_i
  2. Thermalisation process very slow $t \sim e^S$

Eigenstate Thermalisation Hypothesis

- Idea: eigenstates in the microcanonical should window already look thermal
- [Srednicki 1999]:

$$\langle E_i|\mathcal{O}|E_j\rangle = \langle \mathcal{O}\rangle_\beta\, \delta_{ij} + e^{-S(E)/2}\, F_1(E,\omega)\, R_{ij}$$

Random fluctuation

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- It's intrinsically probabilistic

$$\mathbb{E}\Big[\langle E_i|\mathcal{O}|E_j\rangle\Big] = 0 \qquad\qquad \mathbb{E}\left[\big|\langle E_i|\mathcal{O}|E_j\rangle\big|^2\right] = e^{-S(E)}\, F_2(E,\omega)$$

Correlation functions and ETH

- From this point of view, the correlation function becomes a stochastic process

$$G_\beta(t) = \frac{1}{Z(\beta)} \sum_{i,j} e^{-\frac{\beta(E_i+E_j)}{2}} e^{-i(E_i-E_j)t} \left|\langle E_i|\mathcal{O}|E_j\rangle\right|^2$$

Random variable

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Random variable

- This is related to a definition of stochastic processes that is independent of a SDE

Basis of functions

$$X(t) = \sum_n c_n \, f_n(t)$$

Random variable

Moments

- At ‘early’ times

$$\mathbb{E}[G_\beta(t)] \approx \int \mathrm{d}\omega \, e^{-i\omega t} \, F_2(E_\beta, \omega)$$

Moments

- At ‘early’ times

$$\mathbb{E}[G_\beta(t)] \approx \int \mathrm{d}\omega\, e^{-i\omega t}\, F_2(E_\beta, \omega)$$

- To compute the autocorrelation, we need the ansatz

$$\mathbb{E}\left[\left|\mathcal{O}_{ij}\right|^2 \left|\mathcal{O}_{kl}\right|^2\right] = \mathbb{E}\left[\left|\mathcal{O}_{ij}\right|^2\right]\, \mathbb{E}\left[\left|\mathcal{O}_{kl}\right|^2\right] + e^{-2S(E)}\big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\big) H_4(E, \omega)$$

Moments

- Using this ansatz, we find

$$
\begin{aligned}
\mathbb{E}\left[G_\beta(t)G_\beta(s)\right] \approx \mathbb{E}[G_\beta(t)]\ \mathbb{E}[G_\beta(s)] &+ \frac{1}{2\beta\, Z^2(\beta)} \int \mathrm{d}\omega\, e^{-i\omega(t+s)}\, H_4(E_\beta, \omega) \\
&+ \frac{1}{2\beta\, Z^2(\beta)} \int \mathrm{d}\omega\, e^{-i\omega(t-s)}\, H_4(E_\beta, \omega)
\end{aligned}
$$

Moments

- Using this ansatz, we find

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- Same structure that we found in the Brownian case!
- Next: does it make sense to apply this formalism to holographic CFTs?

Holographic CFTs

- Spectrum: gap between the vacuum and Cardy density of states

$$\rho(h,\bar{h}) \approx e^{2\pi\sqrt{\frac{c}{6}\left(h-\frac{c}{24}\right)}+2\pi\sqrt{\frac{c}{6}\left(\bar{h}-\frac{c}{24}\right)}}$$

[Cardy 1986]

[Hartman, Keller, Stoica 2014]

- Asymptotics of structure constants

$$|C_{pqr}|^2 \sim C_0(p,q,r)\,C_0(\bar{p},\bar{q},\bar{r})$$

[Collier, Maloney, Maxfield, Tsiares 2019]

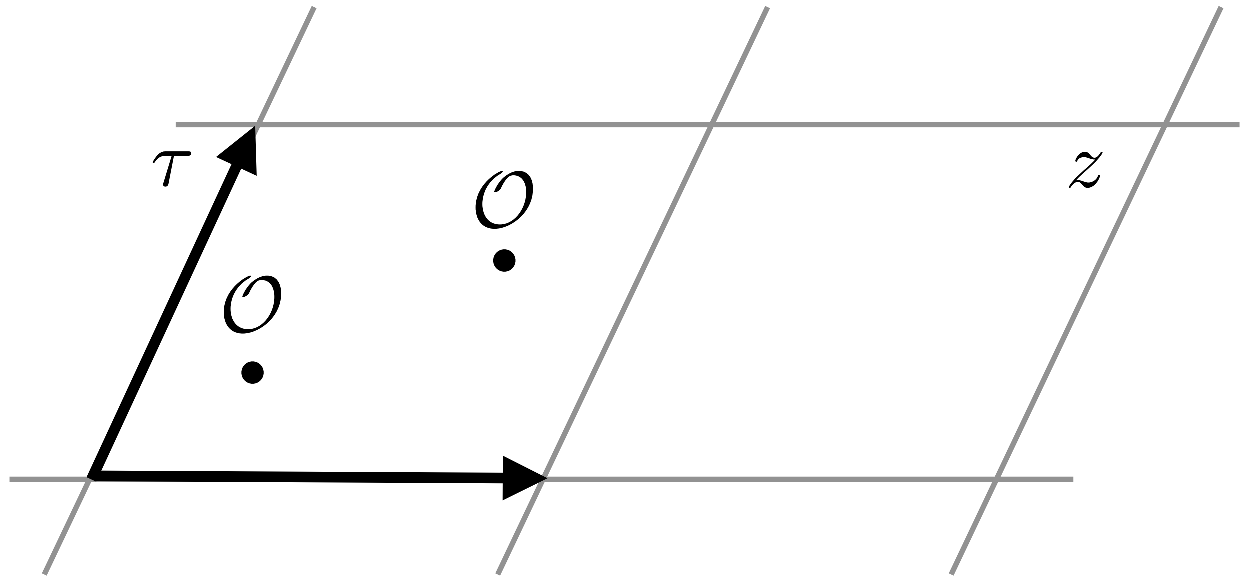
- Thermodynamics: $E_\beta = \dfrac{\pi^2 c}{3\beta^2}$, $S_\beta = \dfrac{2\pi^2 c}{3\beta}$, $\delta E^2 = \langle E^2\rangle_\beta - E_\beta^2 = \dfrac{8\pi^2 c}{3\beta^3}$

very short fluctuations!

Torus correlation functions

- Correlation functions on the torus

$$Z(\tau,\bar\tau) = \mathrm{Tr}\left[e^{2\pi i\tau(L_0-c/24)-2\pi i\bar\tau(\bar L_0-c/24)}\right]$$



$$G_\beta(z,\bar z) = \frac{1}{Z(\tau,\bar\tau)}\,\mathrm{Tr}\left[e^{2\pi i\tau(L_0-c/24)-2\pi i\bar\tau(\bar L_0-c/24)}\,\mathcal{O}(z,\bar z)\,\mathcal{O}(0)\right]$$

- Holographic CFTs and ETH

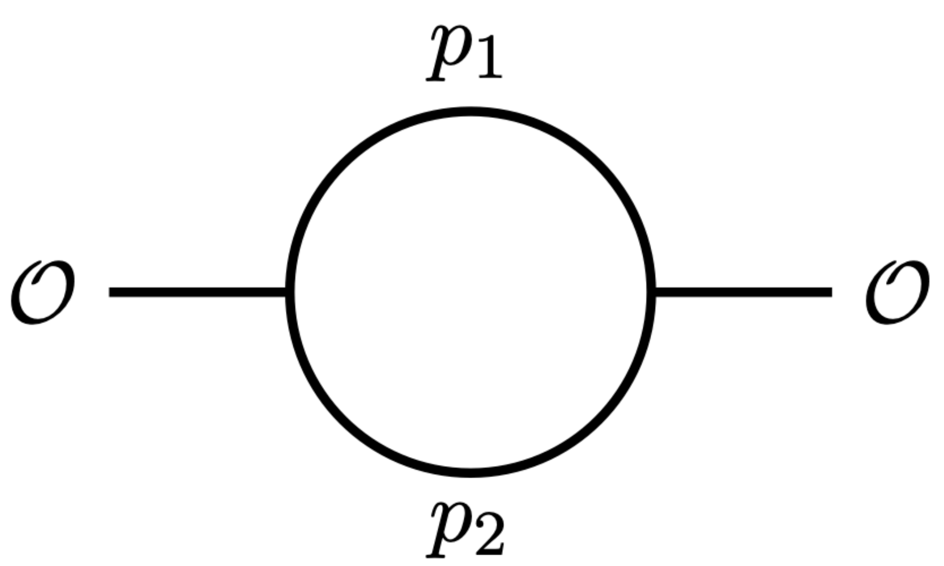
$$\mathbb{E}[C_{pqr}] = 0 \qquad\qquad \mathbb{E}\left[|C_{pqr}|^2\right] = C_0(p,q,r)\,C_0(\bar p,\bar q,\bar r)$$

[Belin, de Boer 2020], [Chandra, Collier, Hartman, Maloney 2022]

Necklace and OPE channel

Necklace

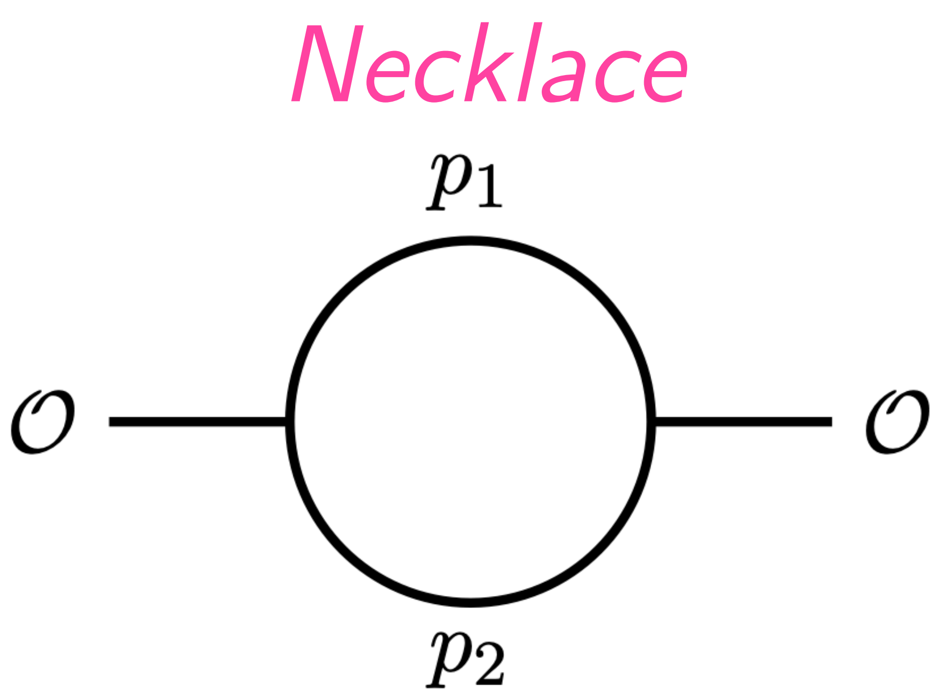
$$G_\beta(z,\bar z) \;=\; \sum_{p_1,p_2}$$



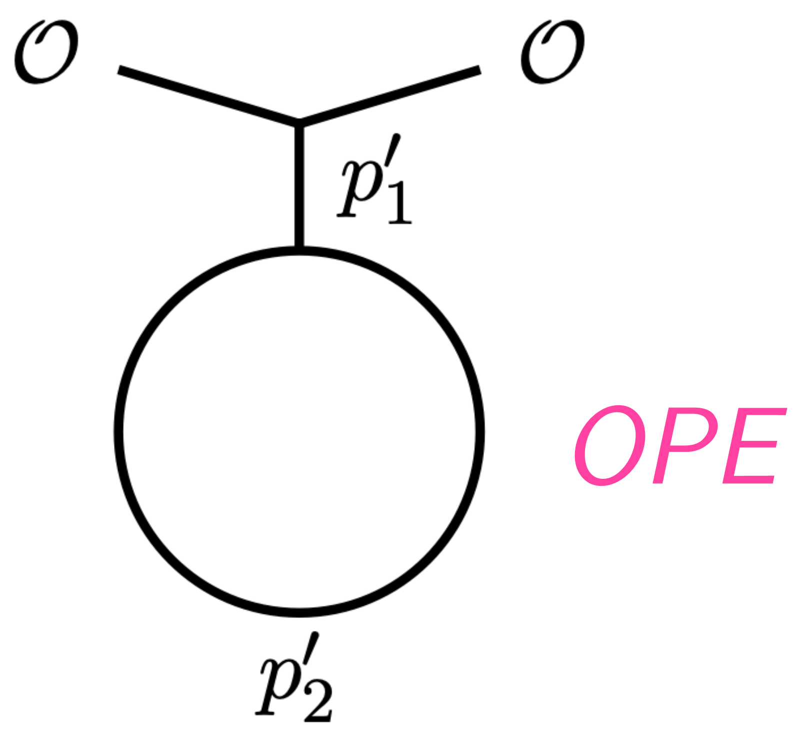
Necklace $$G_\beta(z,\bar z) = \frac{1}{Z(\tau,\bar\tau)} \sum_{p_1,p_2} |C_{\mathcal{O}p_1p_2}|^2\, \mathcal{F}_{\mathrm{N}}(p_1,p_2|\tau,z)\bar{\mathcal{F}}_{\mathrm{N}}(\bar p_1,\bar p_2|\bar\tau,\bar z)$$

Necklace and OPE channel

$$G_\beta(z,\bar{z}) = \sum_{p_1,p_2}$$



$$= \sum_{p_1',p_2'}$$



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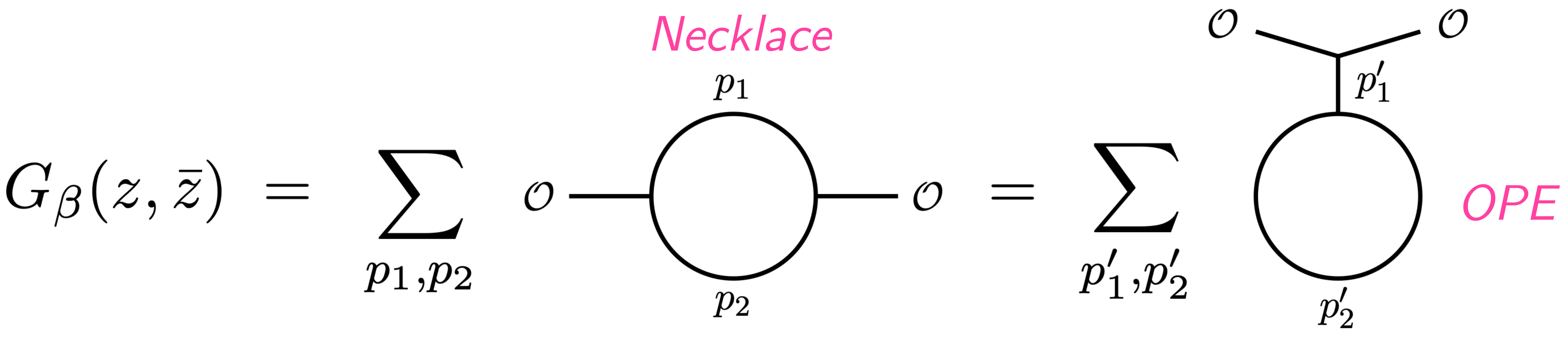
OPE $$G_\beta(z,\bar{z}) = \frac{1}{Z(\tau,\bar{\tau})} \Bigg[\sum_{p'} \mathcal{F}_{\mathrm{OPE}}(1,p'|\tau,v)\,\bar{\mathcal{F}}_{\mathrm{OPE}}(1,\bar{p}'|\bar{\tau},\bar{v}) \; +$$

$$\sum_{p_1',p_2'\neq 1} C_{\mathcal{O}\mathcal{O}p_1'} C_{p_1'p_2'p_2'}\, \mathcal{F}_{\mathrm{OPE}}(p_1',p_2'|\tau,v)\,\bar{\mathcal{F}}_{\mathrm{OPE}}(\bar{p}_1',\bar{p}_2'|\bar{\tau},\bar{v}) \Bigg]$$

Necklace and OPE channel



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Mean

$$\sum_{p_1',p_2'\neq 1} C_{\mathcal{O}\mathcal{O}p_1'} C_{p_1'p_2'p_2'}\, \mathcal{F}_{\rm OPE}(p_1',p_2'|\tau,v)\,\bar{\mathcal{F}}_{\rm OPE}(\bar p_1',\bar p_2'|\bar\tau,\bar v) \Bigg]$$

Stochastic fluctuations

Autocorrelation function in CFTs

- We can also compute the autocorrelation function

$$\mathbb{E}[G_\beta(z,\bar{z})G_\beta(w,\bar{w})] \approx$$

$$\mathbb{E}[G_\beta(z,\bar{z})]\,\mathbb{E}[G_\beta(w,\bar{w})]$$

$$+\ \frac{1}{Z^2(\tau,\bar{\tau})}\,G^L_{(\tau,-\tau)}(z,-w)\,G^L_{(-\bar{\tau},\bar{\tau})}(-\bar{z},\bar{w})$$

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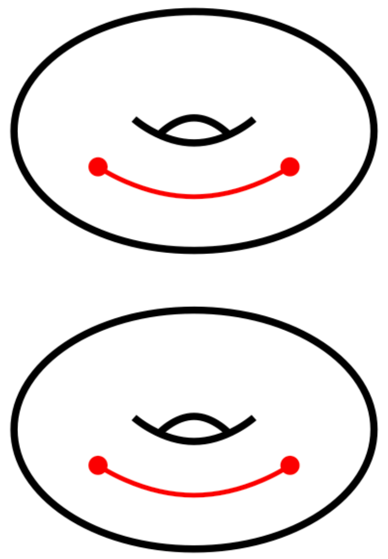

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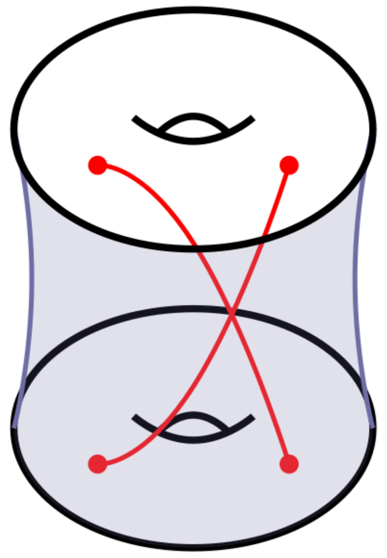

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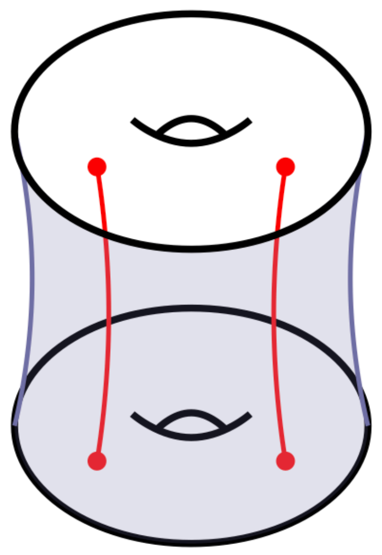

Matches the gravity computation!

[Chandra, Collier, Hartman, Maloney 2022]

Moment vs probability distribution

- From the framework outlined, we can think of semiclassical holography as a mesoscopic duality

AdS		CFT
Moments	$\longleftrightarrow$	*Probability distribution*
$\mathbb{E}[X(t)X(s)\dots]$		$X(t) = \sum_n c_n f_n(t)$

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- In this sense, **gravity is quite unique!** It's the only theory naturally defined in terms of moments, rather than specifying a probability distribution.

Kosambi–Karhunen–Loève Theorem

- **Theorem** (KKL): for any stochastic process $X(t)$, it exists a basis of functions $f_n(t)$ such that the coefficients c_n of the expansion

[Karhunen 1947], [Loeve 1948]

$$X(t) = \sum_n c_n f_n(t)$$

are uncorrelated random variables

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- Our analysis suggests that, for stochastic processes connected to semiclassical gravity, the KKL basis is the set of conformal blocks

A note on ensemble average

- In our framework, an ensemble average is not strictly needed

$$G_\beta(t) = \frac{1}{Z(\beta)} \mathrm{Tr}\left[e^{-\beta H/2}\,\mathcal{O}(t)\,e^{-\beta H/2}\,\mathcal{O}(0)\right]$$

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$$G_\beta(t) = \frac{1}{Z(\beta)}\,\mathrm{Tr}\left[e^{-\beta H/2}\,\mathcal{O}(t)\,e^{-\beta H/2}\,\mathcal{O}(0)\right]$$

$$\updownarrow$$

$$G_E(t) = \langle E|e^{-\beta H/2}\,\mathcal{O}(t)\,e^{-\beta H/2}\,\mathcal{O}(0)|E\rangle$$

- This is very natural from the point of view of stochastic processes and Brownian motion!

Summary

- A mesoscopic description of a system is an intermediate description that contains information on the microscopic theory *in a probabilistic sense*
- Thermal correlation functions in semiclassical gravity in AdS are consistently described by a mesoscopic description
- This phenomenon seems to be a generic features of quantum chaotic systems, relying in particular on ETH
- For holographic CFTs, the KKL expansion is given by conformal blocks

Interesting directions

A stochastic bulk theory?

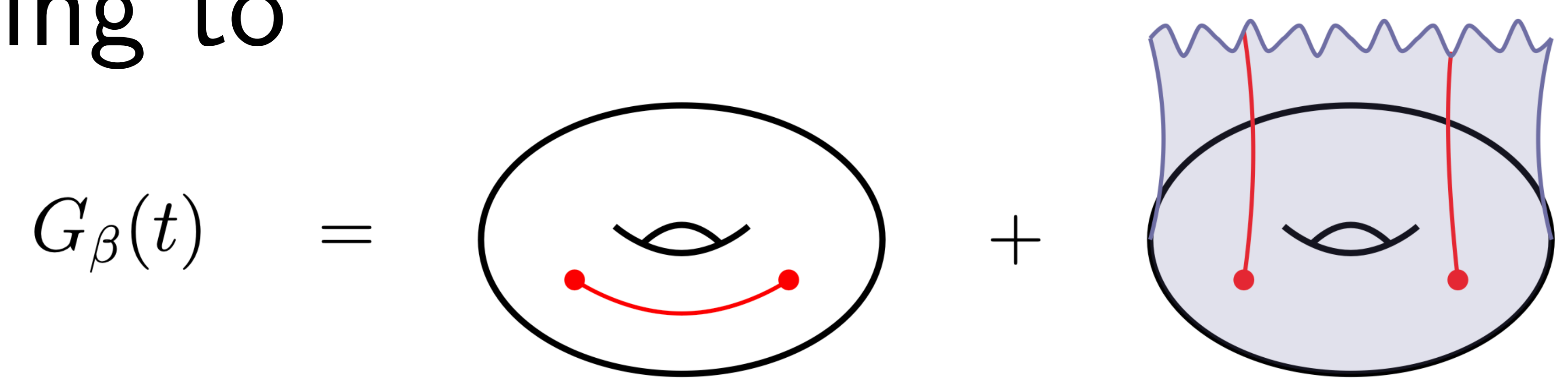
- Can we find a bulk theory that naturally takes into account also wormhole contributions?

- Random tensor models

$$\mathcal{Z} = \int D[L_0, \bar{L}_0]\, D[C]\, e^{-aV(L_0,\bar{L}_0,C)}$$

[Belin, de Boer, Jafferis, Nayak, Sonner 2023]
[Jafferis, Rozenberg, Wong 2024]

- Such a theory needs to give a 'gravitational' meaning to



- Perhaps a path-integral formulation of the stochastic process?

Bootstrapping quantum gravity from quantum chaos?

- Is it possible to give a quantitative estimate of the amount of information is contained in the whole series of moments?
- Two competing answers:
  - Seems we are probing finer and finer quantities
  - Low energy theory still described by a handful of parameters
- We can try to infer properties of the probability distribution
  - *Hamburger Moment Problem*: necessary and sufficient conditions for a series of moments to be described by a positive-definite measure [Lin 2020]
  - Is the probability distribution unique?

Thank you!